On the Existence of Solutions to Boundary Value Problem of Resonance Fourth-order p-Laplace with One Order Derivative

Fei Yang[1, a], YuanJian Lin[2, b]

[1]Nanchang Institute of Science and Technology, Nanchang 330108, Jiangxi

[2]Nanchang Institute of Science and Technology, Nanchang 330108, Jiangxi

[a]feixu126@126.com, [b]linyuanzhou@126.com

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Abstract: This paper deals with the fourth-order p-Laplace boundary value problem of resonance $\begin{cases} (\varphi_p(x''(t)))'' = f(t, x(t), x'(t)), & 0 < t < 1 \\ x(0) = 0, x(1) = ax(\xi), x''(0) = 0, x''(1) = bx''(\eta) \end{cases}$ where $0 < \xi, \eta < 1; a, b > 0$ such that $a\xi = 1$ and $b^{p-1}\eta \le 1$. The existence of solution is obtained by means of Mawhin's continuation theorem.

1. Introduction

Boundary value problems of differential equations are of great significance both in theory and in practice. where, differential equation with p-Laplace operator is an important research field in linear analysis, Many practical problems are translated into the existence of solutions to boundary value problems with p-Laplac operatorse . For example, the application of gas dynamics, research on flight stability of missiles, neuroscience and chemistry [1-3].The study of boundary value problems with p-Laplace operator resonance differential equations can not only improve the basic mathematical theory, but also have an important influence on the study of other disciplines [4-5].

Lu,Jin [6] proved the existence of solutions for the following boundary value problems is studied by using the coincidence degree theory

$$\begin{cases} (\varphi_p(u''(t)))'' = f(t, u(t), u'(t), u''(t)), 0 < t < 1 \\ u(0) = 0, u(1) = au(\xi), u''(0) = 0, u''(1) = bu''(\eta) \end{cases}$$

For this boundary value problem, if $a = b = 0$ and $f(t, u)$ is nonlinear term, By the fixed point theory proved the existence and multiplicity of some solutions [7-8]. In [9], by using the upper and lower solution method proved the existence result of the solution. These studies on boundary value problems are conducted in non-resonant situations. Based on the above results, In this paper, the existence of solutions to the following boundary value problems is studied by using the coincidence degree extension theorem

$$\begin{cases} (\varphi_p(x''(t)))'' = f(t, x(t), x'(t)), & 0 < t < 1 \\ x(0) = 0, x(1) = ax(\xi), x''(0) = 0, x''(1) = bx''(\eta) \end{cases} \tag{1}$$

where $\varphi_p(t) = |t|^{p-2} t$, $f : C([0,1] \times R^2 \to R), 0 < \xi, \eta < 1, a, b > 0$,and $a\xi = b^{p-1}\eta = 1$.

Mawhin's continuation theorem:

Let X, Y be the Banach space, $L : domL \subset X \to Y$ be the Linear mapping, $N : X \to Y$ be the Nonlinear continuous mapping, Let $\dim\ker L = \dim(Y/_{\operatorname{Im} L}) < +\infty$, and $\operatorname{Im} L$ is a Closed set in Y, according to L is a Fredholm operator whose index is zero. If L is a Fredholm operator whose index is zero, then there is a continuous projection operator $P : X \to KerL$ and $Q : Y \to Y_1$, such that $\operatorname{Im} P = K\operatorname{er} L, K\operatorname{er} Q = \operatorname{Im} L, X = KerL \oplus KerP, Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. $L_P := L|_{domL \cap X_1}$ is invertible, so let's call that the inverse K .If $QN(\overline{\Omega})$ is bounded, and $K(I - Q)N : \overline{\Omega} \to X$ is relatively tight in X , according to N is $L-$ tight in $\overline{\Omega}$, where Ω is any bounded open set in X .

Lemma 1.1: (Mawhin coincidence degree theory [10]) Let X, Y be the Banach space, L is a Fredholm operator whose index is zero, $N:\overline{\Omega}\to Y$ is $L-$ tight in $\overline{\Omega}$ .If

(1) $Lx\neq \lambda Nx, \forall (x,\lambda)\in (domL\cap \partial\Omega)\times (0,1)$;

(2) $Nx\in \operatorname{Im} L, \forall x\in KerL\cap \partial\Omega$;

(3) $\deg(JQN, \Omega\cap KerL, 0)\neq 0$, there $J:\operatorname{Im} Q\to KerL$ is a linear isomorphism; equation $Lx=Nx$ has at least one solution in $domL\cap \overline{\Omega}$.

Lemma 1.2[11]: Let $0\le c<\dfrac{1}{\xi}$, if $v\in[0,1]$, BVP $\begin{cases} x''(t)=v(t), t\in(0,1) \\ x(0)=0, x(1)=ax(\xi)\end{cases}$ has a unique solution x, $x(t)=\int_0^1 \Gamma(t,s)v(s)ds, t\in[0,1]$.

$$\Gamma(t,s)=\begin{cases} s\in[0,\xi]: \begin{cases} \dfrac{t}{1-c\xi}[(1-s)-c(\xi-s)], t\le s \\ \dfrac{s}{1-c\xi}[(1-t)-c(\xi-t)], s\le t \end{cases} \\ s\in[\xi,1]: \begin{cases} \dfrac{1}{1-c\xi}t(1-s), \quad t\le s \\ \dfrac{1}{1-c\xi}[s(1-t)+t(t-s)], s\le t \end{cases} \end{cases}$$

Define 1.1: Let $W=\{x: x, \varphi_p(x'')\in C^2[0,1]\}$, if $x\in W$ and satisfies (1), according to x is a solution to the boundary value problem (1).

2. $a\xi=b\eta^{p-1}=1$

When $p\neq 2$, $(\varphi_p(x''))''=(|x''|^{p-2}x'')''$ is nonlinear, so apply the Mawhin's coincidence degree theory, we have BVP (1) in the following form:

$$\begin{cases} u_1''(t)=\varphi_q(u_2(t))=|u_2(t)|^{q-2}u_2(t) \\ u_2''(t)=f(t,u_1(t),u_1'(t)) \\ u_1(0)=0, \quad u_1(1)=au_1(\xi) \\ u_2(0)=0, \quad u_2(1)=b^{p-1}u_2(\eta) \end{cases} \tag{2}$$

There $q>1$ is a constant, and $\dfrac{1}{p}+\dfrac{1}{q}=1$. If $u(t)=(u_1(t),u_2(t))^T$ is a solution to (2), then $u_1(t)$ is a solution to BVP (1).

Define $|\phi|_0=\max_{t\in[0,1]}|\phi(t)|$, $U=\{u=(u_1(\cdot),u_2(\cdot))^T\in C^1[0,1]\times C^1[0,1]\}$, with the norm $\|u\|_U=\max\{|u_1|_0, |u_1'|_0, |u_2|_0, |u_2'|_0\}$; $V=\{v=(v_1(\cdot),v_2(\cdot))^T\in C^1[0,1]\times C^1[0,1]\}$,the norm $\|v\|_V=\max\{|v_1|_0, |v_2|_0\}$. U and V be the Banach space. Let $L: domL\subset U\to V$, and

$$Lu=\begin{pmatrix} u_1'' \\ u_2'' \end{pmatrix} \tag{3}$$

where $domL=\{u\in C^1[0,1]\times C^1[0,1]: u_1(0)=0, u_2(1)=au_1(\xi), u_2(0)=0, u_2(1)=b^{p-1}u_2(\eta)\}$

Let $N:U\to V$, and

$$Nu = \begin{pmatrix} \varphi_q(u_2(t)) \\ f(t,u_1(t),u_1'(t)) \end{pmatrix} \tag{4}$$

so (2) is transformed into an abstract equation: $Lu = Nu, u \in domL$.

It's easy to see by the definition of L, $KerL = \{u = (c_1 t, c_2 t)^T : c_1, c_2 \in R\}$ and $\operatorname{Im} L = \left\{ v = (v_1(\cdot), v_2(\cdot))^T \in V : \int_0^1 \int_{\xi t}^t v_1(s)dsdt = \int_0^1 \int_{\eta t}^t v_2(s)dsdt = 0 \right\}$, Assumed projection operator $P: U \to KerL$ and $Q: Y \to \operatorname{Im} Q$ as follows $(Pu)(t) = (u_1'(0)t, u_2'(0)t)^T$,

$(Qv)(t) = (\frac{2}{1-\xi}\int_0^1 \int_{\xi t}^t v_1(s)dsdt, \frac{2}{1-\eta}\int_0^1 \int_{\xi t}^t v_1(s)dsdt)^T$

For $v \in V$, let $\tilde{v} = v - Qv$, so $\int_0^1 \int_{\xi t}^t \tilde{v}_1(s)dsdt = \int_0^1 \int_{\eta t}^t \tilde{v}_2(s)dsdt = 0$. $\tilde{v} \in \operatorname{Im} L$ and $V = \operatorname{Im} L + R^2$. In addition, $\operatorname{Im} L \cap R^2 = \{0\}$, thus $V = \operatorname{Im} L \oplus R^2$.That means $\dim KerL = co\dim \operatorname{Im} L < +\infty$, so L is an Fredholm operator whose index is zero. On the other hand, K is $L|_{KerP \cap domL}$ inverse, so

$$(Kv)(t) = (\int_0^t \int_0^\tau v_1(s)dsd\tau, \int_0^t \int_0^\tau v_2(s)dsd\tau)^T \tag{5}$$

By (4) and (5), we have N is $L-$tight in $\overline{\Omega}$, Ω is any bounded open set in U.

Theorem 2.1: If $a\xi = b^{p-1}\eta = 1$, and satisfying

(H_1) there is constant $D > 0$ such that $vf(t,u,v) > 0$ （or $vf(t,u,v) < 0$）, for all $|v| > D$, $t \in [0,1]$ and $u \in R$.

(H_2) there is a nonnegative constant r_i, $i = 1,2,3,4,5$ such that

$|f(t,u,v)| \le r_1 |u|^{p-1} + r_2 |v|^{p-1} + r_3$, $(t,u,v) \in [0,1] \times R^2$

when $C_p(r_1 + r_2) < 1$, BVP (1) has at least one solution, where $C_p = \begin{cases} 1, & 1 < p \le 2 \\ 2^{p-1}, & p > 2 \end{cases}$.

Proof: For the equation $Lu = \lambda Nu$, $\lambda \in (0,1)$. Let

$\Omega_1 = \{u \in domL : Lu = \lambda Nu, \lambda \in (0,1)\}$, If $u(t) = (u_1(t), u_2(t))^T \in \Omega_1$, then

$$\begin{cases} u_1''(t) = \varphi_q(u_2(t)) = |u_2(t)|^{q-2} u_2(t) \\ u_2''(t) = f(t,u_1(t),u_1'(t)) \\ u_1(0) = 0, \quad u_1(1) = au_1(\xi) \\ u_2(0) = 0, \quad u_2(1) = b^{p-1}u_2(\eta) \end{cases} \tag{6}$$

First we prove that there is a constant $t_1, t_2 \in [0,1]$, such that

$$|u_1'(t_1)| \le D \tag{7}$$

$$u_2(t_2) = 0 \tag{8}$$

In fact, by $Lu = \lambda Nu$, we get $QNx = 0$, thus $\int_0^1 \int_{\eta t}^t f(s,u_1(s),u_1'(s)))dsdt = 0$, so, there are $t_1 \in [0,1]$, such that $f(t_1,u_1(t_1),u_1'(t_1)) = 0$, by ($H_1$), we get (7) set up.

On the other hand, by boundary conditions and functions $u_1(t)$ is continuous in $[0,1]$, we get $\xi_1 \in (0,\xi)$ 和 $\xi_2 \in (\xi,1)$, such that $au_1(\xi) - u_1(\xi) = (a-1)[u_1(\xi) - u_1(0)] = (1-\xi)u_1'(\xi_1)$, $u_1(1) - u_1(\xi) = (1-\xi)u_1'(\xi_2)$; there are $u_1'(\xi_1) = u_1'(\xi_2)$, $\xi_3 \in (\xi_1,\xi_2) \subset (0,1)$, such that $u_1''(\xi_3) = 0$, then $u_2(\xi_3) = \varphi_p(u_1''(\xi_3)) = 0$ .by $u_2(0) = 0$ and $u_2(t)$ is continuous in $[0,1]$, we gen (8) set up.

The second, by (6) and (H_2), we get

$$\int_0^1 \left|u_2''(t)\right|dt = \lambda\int_0^1 \left|f(t,u_1(t),u_1'(t))\right|dt$$
$$\le r_1\int_0^1 |u_1(t)|^{p-1}dt + r_2\int_0^1 \left|u_1'(t)\right|^{p-1}dt + r_3$$
$$\le r_1|u_1|_0^{p-1} + r_2\left|u_1'\right|_0^{p-1} + r_3 \quad (9)$$

By (7)(8) and *Hölder* inequality, we have

$$|u_1|_0 \le \left|\int_0^t u_1'(s)ds\right| \le \left|u_1'\right|_0 \le \left|u_1'(t_1) + \int_{t_1}^t u_1''(s)ds\right| \le D + \int_0^1 \left|u_1''(s)ds\right| \quad (10)$$

$$|u_2|_0 \le \left|\int_0^t u_2'(s)ds\right| \le \left|u_2'\right|_0 \le \left|u_2'(t_2) + \int_{t_2}^t u_2''(s)ds\right| \le \int_0^1 \left|u_2''(s)\right|ds \quad (11)$$

By (6), we get $\int_0^1 \left|u_1''(s)\right|ds = \lambda\int_0^1 \left|\varphi_q(u_2(t))\right|ds \le \varphi_q(|u_2|_0)$ (12)

Substitute equation (10-12) into equation (9), we get

$$\int_0^1 \left|u_2''(t)\right|dt \le (r_1+r_2)(D+\varphi_q(|u_2|_0))^{p-1} + r_3$$
$$\le C_p(r_1+r_2)(D^{p-1} + |u_2|_0) + r_3$$
$$\le C_p(r_1+r_2)\int_0^1 \left|u_2''(t)\right|dt + r_3 \quad (13)$$

by $p>1$ and $C_p(r_1+r_2)+r_3+r_4<1$ set up, The above formula indicates that there is a constant $M_1>0$, such that $\int_0^1 \left|u_2''(t)\right|dt \le M_1$

so, $|u_2|_0 \le \left|u_2'\right|_0 \le M_1$ (14)

$$|u_1|_0 \le \left|u_1'\right|_0 \le D + M_1^{q-1} := M_2 \quad (15)$$

Let $\Omega = \left\{u\in U : \|u\|_U < \max\left\{M_1,M_2\right\}+1\right\}$, The lemma 1.1 condition (1) is satisfied. Without loss of generality, Assuming that $|v|>D, t\in[0,1]$ and $u\in R$, $vf(t,u,v)>0$ is set up. So let's prove that for $u\in KerL\cap\partial\Omega$ 有 $Nu\notin \operatorname{Im}L$. Otherwise, there are $u_0=(c_1t,c_2t)\in KerL$ such that $Nu_0=(\varphi_q(u_2), f(t,c_1t,c_1)\in \operatorname{Im}L$ .That is $QNu_0=0$, so $\int_0^1\int_{\eta t}^t f(s,c_1s,c_1)dsdt=0$ .According to the condition (H_1),we get $|c_1|\le D<M_2$, That contradicts $u_0\in\partial\Omega$. Therefore, condition (2) in lemma 1.1 is also satisfied.

Let the mapping $J:\operatorname{Im}Q\to KerL$ is $J(c_1,c_2)=(c_1t,c_2t)$, and $H(u,\mu)=\mu u+(1-\mu)JQN, \forall(u,\mu)\in\overline{\Omega}\times[0,1]$; For $u\in(\partial\Omega\cap KerL)\times[0,1]$,we have

$$H(u,\mu)=\begin{pmatrix}\mu c_1t+\dfrac{2(1-\mu)}{1-\eta}\int_0^1\int_{\eta t}^t f(s,c_1s,c_1)dsdt \\ \mu c_2t+\dfrac{2(1-\mu)}{1-\xi}\int_0^1\int_{\xi t}^t \varphi_q(c_2s)dsdt\end{pmatrix}\ne 0$$

So, $\deg\left\{JQN,\Omega\cap KerL,0\right\}=\deg\left\{H(0,u),\Omega\cap KerL,0\right\}$
$=\deg\left\{H(1,u),\Omega\cap KerL,0\right\}=\deg\left\{I,\Omega\cap KerL,0\right\}\ne 0$,

That is, condition (3) in lemma 1.1 is satisfied. According to lemma 1.1, there is a solution to equation (6) in $\overline{\Omega}\cap domL$ $u^*(t)=(u_1^*(t),u_2^*(t))^T$ .so BVP（1）has a solution $u_1^*(t)$.

3. Conclusion

In this paper, the existence of at least one solutions to boundary value problem of resonance fourth-order p-Laplace with one order derivative is considered; By means of Mawhin's continuation theorem, the existence of solution is verified .

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